

Selberg Sums – a new perspective

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Abstract

Selberg sums are the analogues over finite fields of certain integrals studied by Selberg in in 1940s. The original versions of these sums were introduced by R.J.Evans in 1981 and, following an elegant idea of G.W.Anderson in 1991 they were evaluated by Anderson, Evans and P.B. van Wamelen. In 2007 the author noted that these sums and certain generalizations of them appear in the study of the distribution of Gauss sums over a rational function field over a finite field. The distribution of Gauss sums is closely related to the distribution of the values of the discriminant of polynomials of a fixed degree. Here we shall take this up further. The main goal here is to establish the basic properties of Selberg sums and to formulate the problems which arise from this point of view.

1 Introduction

The usual class of character sums known as Selberg sums was introduced in 1981 by R.J.Evans, [5]. They were the analogues over finite fields of a class, or rather several related classes, of integrals introduced by A. Selberg in 1944, [12, p.204 ff.]. In fact he had already used an integral of this type in 1941 in [12, p.74 ff.] but was uncertain as to whether the integral was already known and waited before publishing details.

Characteristic of these integrals is that they are taken over a space of monic polynomials of a fixed degree and that one factor in the integrand is a power of the discriminant. Selberg regarded the integrals he studied as a extension of Euler's beta-function and, indeed, his evaluation gives the integrals as a quotient of products of gamma functions. The primary example is the integral

$$\int_0^1 \cdots \int_0^1 (x_1 \cdots x_i)^{\alpha-1} ((1-x_1) \cdots (1-x_n))^{\beta-1} \prod_{i < j} (x_i - x_j)^{2\gamma} dx_1 \cdots dx_i$$

in the region $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, $\operatorname{Re}(\gamma) > -\operatorname{Min}(\frac{1}{i}, \frac{\operatorname{Re}(\alpha)}{i-1}, \frac{\operatorname{Re}(\beta)}{i-1})$. Selberg's evaluation shows that the integral is equal to

$$\prod_{j=1}^i \frac{\Gamma(1+j\gamma)\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(i+j-2)\gamma)}.$$

For our purposes a transformation is helpful. This integral can be regarded as over all monic polynomials of degree i with all their roots in $[0, 1]$. We let $\sigma_1, \dots, \sigma_i$ be the standard symmetric functions in x_1, \dots, x_i . Then one has

$$d\sigma_1 \wedge d\sigma_2 \wedge \dots \wedge d\sigma_i = \prod_{j < j'} (x_j - x_{j'}) dx_1 \wedge dx_2 \wedge \dots \wedge dx_i.$$

It follows that we can write Selberg's integral as

$$\int |f(0)|^\alpha |f(1)|^\beta |D(f)|^{\gamma - \frac{1}{2}} df,$$

where the integral is over the set of polynomials described above and df is the standard Lebesgue measure on the affine space of the f .

We can now describe Evans' analogue. Let q be a power of the odd prime number p . Let χ_1, χ_2 and χ_3 be multiplicative characters on \mathbb{F}_q^\times and let ω be the unique quadratic character. Then Evans introduces

$$\sum \chi_1(f(0))\chi_2(f(1))(\omega\chi_3)(D(f))$$

where $D(f)$ denotes the discriminant of the polynomial f and the sum is taken over all monic polynomials of a fixed degree i . It turns out that introducing the factor ω makes the development much smoother and it can be regarded as the analogue of the term $-\frac{1}{2}$ in the exponent of $D(f)$ in the alternative version of Selberg's integral.

In 1990 G.W. Anderson [1] found an ingenious and remarkably simple method for evaluating some of these sums. Following this first R.J. Evans, [6] and then P.B. v. Wamelen [15] extended Anderson's method to cover all cases. Details of the results are given in §3.

Since Selberg's integral can be considered as an extension of the beta-integral we can regard Evans' sum as an extension of their analogues over finite fields, namely Jacobi sums. One immediate generalization of the beta-integral is the standard integral representation of the hypergeometric function and their generalizations. In [10] it was pointed out that the finite field analogues of these functions intervene in the theory of metaplectic forms, and especially of Eisenstein series, over rational function fields over finite fields.

In this article we shall follow this line of thought and apply the appellation “Selberg sum” to this more general class of sum. To the best of my knowledge the corresponding archimedean functions have not been investigated.

The theory of metaplectic forms leads us to the evaluation to some more general Selberg sums. This is an aspect which we shall not go into here; some indications are given in [11]. Here we shall take up a theme of that paper, namely a transformation formula. This will be given in Section 4. It is quite elementary but among other things it shows that the results of Anderson, Evans and v.Wamelen can be used to evaluate further interesting classes of Selberg sums.

It is worth noting that it is relatively easy to compute specific examples of Selberg sums and so one can investigate them experimentally. We shall explain why this is so later. This means that one can investigate the sums experimentally and, for example, examine their sizes in various metrics. This is a topic which we shall postpone to a later paper.

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2 Notations

Let, as before, q be a power of the odd prime q . For $f, g \in \mathbb{F}_q[x]$ let $R(f, g)$ be the resultant of $R(f, g)$. We recall that $R(f, g)$ is a universal polynomial (Sylvester’s determinant) in the coefficients of f and g . The function $g \mapsto R(f, g)$ depends only on $g \pmod{f}$. Also $R(f, g)$ is bimultiplicative and satisfies the reciprocity law

$$R(g, f) = (-1)^{\deg(f)\deg(g)} R(f, g).$$

Finally $R(x - a, f) = f(a)$. These properties determine R completely and can be cast into the form of an efficient algorithm based on the model of continued fractions for its evaluation.

For $f \in \mathbb{F}_q[x]$ let $D(f)$ be the discriminant of f . If f is monic one has $D(f) = \eta(\deg(f))R(f, f')$ where $\eta(i) = 1$ for $i \equiv 0, 1 \pmod{4}$ and $\eta(i) = -1$ for $i \equiv 2, 3 \pmod{4}$ ¹.

We let μ be the Möbius function on $\mathbb{F}_q[x]$. Then Pellet’s formula (see [2, (1.5)], [4, Lemma 4.1] or [13]) asserts that

$$\mu(f) = (-1)^{\deg(f)} \omega(D(f))$$

¹Note that this factor is missing from the formula (5) on [14, p.87], an oversight which unfortunately led to author to similar inaccuracies in [10].

where, as above, ω is the unique quadratic character on \mathbb{F}_q^\times .

Let X_q be the group of characters on \mathbb{F}_q^\times which we consider as taking values in $\mathbb{Q}(1^{\frac{1}{q-1}})$. For $\chi \in X_q$ let $\text{ord}(\chi)$ denote the order of χ . Let e_o be a non-trivial additive character on \mathbb{F}_q ; for our purposes it is convenient to regard it as taking values in $\mathbb{Q}(1^{\frac{1}{p}})$. The choice of e_o will not play any significant role in our discussion. For $\chi \in X_q$ we define the Gauss sum over \mathbb{F}_q to be

$$\tau(\chi) = \sum_{a \in \mathbb{F}_q^\times} \chi(a) e_o(a).$$

For $\chi_1, \chi_2 \in X_q$ let

$$J(\chi_1, \chi_2) = \sum_{\substack{a \in \mathbb{F}_q \\ a \neq 0, 1}} \chi_1(a) \chi_2(1-a)$$

be the Jacobi sum. Then if $\chi_1 \chi_2 \neq 1$

$$\tau(\chi_1) \tau(\chi_2) = J(\chi_1, \chi_2) \tau(\chi_1 \chi_2)$$

whereas if $\chi_1 \chi_2 = 1$, but $\chi_1, \chi_2 \neq 1$ then

$$\tau(\chi_1) \tau(\chi_2) = \chi_1(-1) q$$

and

$$J(\chi_1, \chi_2) = -\chi_1(-1).$$

Additionally one has $J(\chi, 1) = -1$ if $\chi \neq 1$ and $J(1, 1) = q - 2$. Also $J(\chi_1, \chi_2) = J(\chi_2, \chi_1)$, $J(\chi_1, \chi_2) = \chi_2(-1) J((\chi_1 \chi_2)^{-1}, \chi_2)$ and $J(\chi_1, \chi_2) = \chi_1(-1) J(\chi_1, (\chi_1 \chi_2)^{-1})$. There are some further relations and properties of these sums which have been studied in great detail.

Let now $k = \mathbb{F}_q(x)$ and $R = \mathbb{F}_q[x]$. We define

$$e : k \rightarrow \mathbb{Q}(1^{\frac{1}{p}})$$

by

$$\begin{aligned} e(f) &= e_o(\sum_{v \neq \infty} \text{Res}_v(f dx)) \\ &= e_o(-\text{Res}_\infty(f dx)). \end{aligned}$$

Here, in the sum, v runs through the finite places of k .

We define for $\chi \in X_q$ and $g \in R - \{0\}$ the Dirichlet character of modulus g

$$f \mapsto \chi(f/g) = \chi(R(g, f)).$$

We define the global Gauss sum associated with k to be

$$g(r, \chi, c) = \sum_{d \pmod{c}} \chi(d/c) e(rd/c)$$

for $r \in R - \{0\}$. This can, by means of the Davenport-Hasse theorem, be evaluated in terms of Gauss sums over \mathbb{F}_q . If r is coprime to c one obtains

$$g(r, \chi, c) = \mu(c) \chi(r/c)^{-1} \chi(c'/c) (-\tau(\chi))^{\deg(c)}.$$

Here c' denotes the derivative of c . This shows that the function $\chi(c'/c)$ is as subtle a function of c as $g(r, \chi, c)$. Recall that it is equal to $\chi(\eta(\deg(c)))\chi(D(c))$ for c monic. We therefore have for c coprime to r

$$g(r, \chi, c) = \mu(c) \chi(\eta(\deg(c))) \chi(r/c)^{-1} (-\tau(\chi))^{\deg(c)} \chi(D(c)).$$

In the special case $\chi = \omega$, $r = 1$ we obtain

$$g(r, \omega, c) = \mu(c)^2 \omega(\eta(\deg(c))) \tau(\omega)^{\deg(c)}.$$

As $\tau(\omega)$ is \sqrt{q} times a fourth root of 1 this gives us the Gauss evaluation of the quadratic Gauss sum in this context. One could also use this argument to prove Pellet's formula using that evaluation, as given, for example, in [16, XIII, §11, Proof of Theorem 13].

We can now define the Selberg sums which we are going to investigate here. Apart from a factor ± 1 it specializes to the one used in the papers of Anderson, Evans and v. Wamelen when we take r to be of the form $x^{e_0}(x - 1)^{e_1}$. For χ_1, χ_2 it is

$$\sum_c \chi_1(r/c) \omega(D(c)) \chi_2(D(c))$$

where c is summed over all monic polynomials of a fixed degree i . For our purposes it is convenient to modify the expression by using Pellet's formula. It becomes

$$(-1)^i \sum_c \mu(c) \chi_1(r/c) \chi_2(D(c)),$$

or, with the reciprocity law,

$$(-1)^i \chi(-1)^{i \deg(r)} \sum_c \mu(c) \chi_1(c/r) \chi_2(D(c)).$$

Both of the sums here could be considered as to be entitled to be the more fundamental one. We choose

$$Se(r, \chi_1, \chi_2, i) = \sum_{\substack{c \text{ monic} \\ \deg(c)=i}} \mu(c) \chi_1(r/c) \chi_2(D(c)).$$

We take $\chi_1(r/c)$ to be zero if r and c are not coprime. By means of the Davenport-Hasse theorem we have then

$$Se(r, \chi_1, \chi_2, i) = (-1)^i \tau(\chi_2)^{-i} \chi_2(\eta(i)) \sum_{\substack{c \text{ monic} \\ \deg(c)=i}} \chi_1(r/c) g(1, \chi_2, c).$$

If now there exists an exponent e so that

$$\chi_1 \chi_2^e = 1$$

then

$$Se(r, \chi_1, \chi_2, i) = (-1)^i \tau(\chi_2)^{-i} \chi_2(\eta(i)) \sum_{\substack{c \text{ monic} \\ \deg(c)=i}} g(r^e, \chi_2, c)$$

where we restrict the sum to c coprime to r . In this case one can use of the theory of metaplectic forms to investigate this sums, see [10],[11]. At the other extreme $\chi_2 = 1$ and $Se(r, \chi_1, 1, i)$ is then the coefficient of q^{-is} in the L-series $L(s, \chi_1(r/\cdot))$ (over R). It follows from this that if $\chi_1(r/\cdot)$ is non-principal then $Se(r, \chi_1, 1, i) = 0$ for $i \geq \deg(r_0)$ where r_0 is the conductor of $\chi_1(r/\cdot)$ – [1, p.471].

We should note that for r of the form θr_o^a with $\theta \in \mathbb{F}_q^\times$, $a \in \mathbb{N}$ one has

$$Se(r, \chi_1, \chi_2, i) = \chi_1(\theta)^i Se(r_o, \chi_1^a, \chi_2, i).$$

In particular we can reduce the calculation of $Se(r, \chi_1, \chi_2, i)$ to the case where r is monic.

3 The Anderson-Evans-v. Wamelen evaluation of Selberg sums

In this section we shall summarize the results of Anderson, Evans and v. Wamelen in a notation suitable for the purposes in hand. We shall take $r(x) = x^{e_0}(x-1)^{e_1}$. The group X_q is cyclic and the notation of these three authors is based on the choice of a generator denoted by τ . The parameters a, b and c are related to the parameters here by $\tau^a = \chi_1^{e_0}$, $\tau^b = \chi_1^{e_1}$ and $\tau^c = \chi_2$. It is convenient to distinguish two cases. If $\tau^a = \chi_1^{e_0}$ and $\tau^b = \chi_1^{e_1}$ lie in the subgroup generated by $\tau^c = \chi_2$ then we can define two integers f_0 and f_1 by $\chi_1^{e_0} \chi_2^{f_0} = 1$ and $\chi_1^{e_1} \chi_2^{f_1} = 1$. It turns out that it is this case, the one investigated by v. Wamelen in [15], is precisely the one covered by the theory of metaplectic forms as formulated in [11]. We shall therefore refer to this as the *metaplectic* case. The alternative, when either $\chi_1^{e_0}$ or $\chi_1^{e_1}$ is

not in the subgroup of X_q generated by χ_2 we call *non-metaplectic*. This was the one investigated by Evans in [6]. This nomenclature is by no means satisfactory, nor even accurate, but it has developed from my private usage and I have not found anything better to propose. The distinction does seem to be a useful one as we shall see below.

In the discussion below we shall write i instead of n as used by Anderson, Evans and v.Wamelen. This corresponds to the usage in Section 2. We shall also use n , the order of χ_2 , where they used d . For $m \in \mathbb{N}$ and $j \in \mathbb{Z}$ we write $(j)_m$ for the least non-negative residue of j modulo m .

In the theory of Selberg sums a particularly important role is played by a product denoted by previous authors by $P_n(a, b, c)$ and which we shall replace by

$$P_i(e_0, e_1, \chi_1, \chi_2) = \prod_{0 \leq j < i} \frac{\tau(\chi_1^{e_0} \chi_2^j) \tau(\chi_1^{e_1} \chi_2^j) \tau(\chi_2^{j+1}) \overline{\tau(\chi_1^{e_0+e_1} \chi_2^{i-1+j})}}{q \tau(\chi_2)}.$$

This product corresponds to Selberg's product of gamma functions. It is clear that for a certain A one has

$$P_{i+\ell n}(e_0, e_1, \chi_1, \chi_2) = A^\ell P_i(e_0, e_1, \chi_1, \chi_2).$$

One can express A in a simple form, [6, Lemma 3.1],

$$\begin{aligned} A &= q^{n-2} \chi_2(\eta(n)) / \tau(\chi_2)^n && \text{in the metaplectic case} \\ &= -J(\chi_1^{e_0 n}, \chi_1^{e_1 n}) q^{n-1} \chi_2(\eta(n)) / \tau(\chi_2)^n && \text{in the non-metaplectic case.} \end{aligned}$$

Note that in the latter case at least one of $\chi_1^{e_0 n}$ and $\chi_1^{e_1 n}$ is non-trivial. Our notations are the reason that this looks a little different from the formulation in [6].

It is worth-while noting that in [15, Lemma 2] v. Wamelen proves the additional evaluation that $P_{f_0+f_1+1}(e_0, e_1, \chi_1, \chi_2)$ is equal to

$$-q^{(f_0+f_1)n-1} \chi_1(-1)^{f_0+f_1+1} \chi_2(\eta(f_0+f_1+1)) \frac{\tau(\chi_2^{f_0+f_1+1})}{\tau(\chi_2)^{f_0+f_1+1}}.$$

These evaluations are consequences of the Davenport-Hasse analogue of the Gauss-Legendre multiplication formula for the gamma function.

For $y \in \mathbb{N}$ let

$$T(y, q) = -y + \sum_{k=0}^y (2k+1)(-q)^{y-k}$$

and

$$S(y, q) = 1 - (1 - q)y.$$

From now on we shall assume the $n > 1$ as Anderson's argument has to be modified in the exceptional case $n = 1$. For more details see §5. The main result of the three authors is that in the non-metaplectic case

$$Se(x^{e_0}(x-1)^{e_1}, \chi_1, \chi_2, i)$$

is equal to

$$\chi_1(-1)^{e_1 i}(-1)^i P_i(e_0, e_1, \chi_1, \chi_2)$$

and in the metaplectic case it is equal to

$$\chi_1(-1)^{e_1 i}(-1)^i q^{\lfloor \frac{i}{n} \rfloor} P_i(e_0, e_1, \chi_1, \chi_2)$$

times

$$\begin{array}{ll} T(2 \lfloor \frac{i}{n} \rfloor, q) & \text{if } (i)_n \leq \text{Min}(f_0, f_1) \leq \text{Max}(f_0, f_1) < (f_0 + f_1 - i + 1)_n \\ T(2 \lfloor \frac{i}{n} \rfloor + 1, q) & \text{if } (f_0 + f_1 - i + 1)_n \leq \text{Min}(f_0, f_1) \leq \text{Max}(f_0, f_1) < (i)_n \\ S(\lfloor \frac{i}{n} \rfloor + 1, q) & \text{otherwise.} \end{array}$$

We note here, following [15, §4], the following summations,

$$\sum_{m \geq 0} T(2m, q) X^m = \frac{U_e(q, X)}{(1 - X)^2(1 - q^2 X)}$$

where

$$U_e(q, X) = 2X^2 q^2 - qX^2 - 3qX + X + 1,$$

$$\sum_{m \geq 0} T(2m + 1, q) X^m = \frac{U_o(q, X)}{(1 - X)^2(1 - q^2 X)}$$

where

$$U_o(q, X) = X^2 q^2 + q^2 X - 3qX - q + 2,$$

and

$$\sum_{m \geq 0} S(m, q) X^m = \frac{1 + (q - 2)X}{(1 - X)^2}.$$

We note also that $U_e(q, q^{-2}) = (1 - q^{-1})^3$, $U_e(q, 1) = 2(q - 1)^2$, $U_o(q, q^{-2}) = -q(1 - q^{-1})^3$ and $U_o(q, 1) = 2(q - 1)^2$. These results mean that in the non-metaplectic case and in any complex embedding we have

$$\sum_{\substack{i \geq 0 \\ i \equiv i_0}} S(x^{e_0}(x-1)^{e_1}, \chi_1, \chi_2, i) X^{(i-i_0)/n}$$

is equal to

$$\chi_1(-1)^{e_1 i_0} (-1)^{i_0} P_{i_0}(e_0, e_1, \chi_1, \chi_2) / (1 - \chi_1(-1)^{e_1 n} (-1)^n AX)$$

where we assume that $0 \leq i_0 < n$ and A is as above. Note that $|A|$ takes on, if $n \neq 1$, one of the three values $q^{\frac{n}{2}-1}$, $q^{\frac{n}{2}-\frac{1}{2}}$ or $q^{\frac{n}{2}}$ in any complex embedding. If $n = 1$ then $|A|$ takes on one of the three values 1 , $q^{\frac{1}{2}}$ or q in any complex embedding.

In the metaplectic case we find that, according to the three cases above, the series is equal to

$$\begin{aligned} & \chi_1(-1)^{e_1 i_0} (-1)^{i_0} \frac{P_{i_0}(e_0, e_1, \chi_1, \chi_2) U_e(q, \chi_1(-1)^{e_1 n} (-1)^n AX)}{(1 - \chi_1(-1)^{e_1 n} (-1)^n q AX)^2 (1 - \chi_1(-1)^{e_1 n} (-1)^n q^3 AX)}, \\ & \chi_1(-1)^{e_1 i_0} (-1)^{i_0} \frac{P_{i_0}(e_0, e_1, \chi_1, \chi_2) U_0(q, \chi_1(-1)^{e_1 n} (-1)^n) AX}{(1 - \chi_1(-1)^{e_1 n} (-1)^n q AX)^2 (1 - \chi_1(-1)^{e_1 n} (-1)^n q^3 AX)}, \end{aligned}$$

or

$$\chi_1(-1)^{e_1 i_0} (-1)^{i_0} \frac{P_{i_0}(e_0, e_1, \chi_1, \chi_2) (1 + (q - 2) \chi_1(-1)^{e_1 n} (-1)^n q AX)^2}{(1 - \chi_1(-1)^{e_1 n} (-1)^n q AX)^2}$$

respectively. Note that the nature of the singularities reflect the type of the Selberg sum. As we have assumed that $n > 1$ we have in the metaplectic case $|A| = q^{\frac{n}{2}-2}$.

4 The transformation formula

We shall now turn to a property of Selberg sums analogous to Theorem 1 of [11]. We need some preparations in order to be able to formulate the result. Let n be the order of χ_2 and let n' be the order of χ_1 . Let χ_0 be such that χ_1 and χ_2^2 are in the group generated by χ_0 . Let $a \geq 0$ and $b \geq 0$ be such that

$$\chi_1 = \chi_0^a, \quad \chi_2^2 = \chi_0^b.$$

Let π monic and a prime in $R = \mathbb{F}_q[x]$. If $\pi | r$ then

$$Se(\pi^{n'} r, \chi_1, \chi_2, i) = Se(r, \chi_1, \chi_2, i).$$

If $\pi \nmid r$ this is no longer true. We have that

$$Se(r, \chi_1, \chi_2, i) - Se(\pi^{n'} r, \chi_1, \chi_2, i)$$

is equal to

$$\sum_{\deg(c_1 \pi) = i} \mu(c_1 \pi) \chi_1(r/c_1 \pi) \chi_2(D(c_1 \pi)).$$

Since $D(c_1\pi) = D(c_1)D(\pi)R(c_1, \pi)^2$ this becomes

$$-\chi_1(r/\pi)\chi_2(D(\pi)) \sum_{\deg(c_1)=i-\deg(\pi)} \mu(c_1)\chi_1(r/c_1\pi)\chi_2(\pi/c_2)^2\chi_2(D(c_1))$$

or

$$-\chi_1(r/\pi)\chi_2(D(\pi))Se(r^a\pi^b, \chi_0, \chi_2, i - \deg(\pi)).$$

We regard this as a stability property of Selberg sums.

Theorem 1 *Let $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$ be such that $\Delta = \alpha\delta - \beta\gamma \neq 0$. Then for suitable integers M, M' we have that*

$$\chi_2(\Delta)^{1-i}Se\left(r\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)^a\left(\frac{1}{(\gamma x + \delta)^2}\right)^{b(i-1)+M}, \chi_0, \chi_2, i\right)$$

is equal to

$$Se(r(x)(-\gamma x + \alpha)^{M'}, \chi_1, \chi_2, i)$$

If $\gamma = 0$ we can take $M, M' = 0$. If $\gamma \neq 0$ then M is to such that $\chi_0^M = 1$ and $M + b(i-1) > a \cdot \deg(r)$, and M' is such that $M' > 0$ and $\chi_1^{M'} = 1$.

Proof: The proof is carried out by verifying the identity for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of the form $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ and equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and then combining these in the usual manner.

The first two cases are straightforward; we need only replace c in the sum defining the Selberg sum by $c(x+\lambda)$ and $c(\theta x)\theta^{-i}$ where $\theta = \alpha/\delta$ respectively.

In the case $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we assume that $r(0) = 0$; this is the reason for the introduction of the parameter M' . This being so c is not divisible by x and so $c(0) \neq 0$. We replace c in the sum by $c(x^{-1})x^i/c(0)$ and recall that $R(x, c) = c(0)$. The formula quoted now follows from the theory of resultants.

5 Selberg sums and metaplectic groups

The results of Section 3 show that there is a large number of relationships between Selberg sums. The results of Anderson, Evans and v. Wamelen show that polynomials with two rational zeros over \mathbb{F}_q can be evaluated explicitly. It turns out, as we shall see in the next section, that if the zeros are no longer rational a similar formula holds. The next case which one can investigate, as in [11] is that of polynomials with three rational zeros. In view of Theorem

1 these can be brought to the form $x^{e_0}(x-1)^{e_1}(x-\lambda)^{e_\lambda}$. As in [11] there are a large number of relations between these, now a little more complicated as we have to move into the region of stability. They are the analogues of the relations between such sums similar to transformations of hypergeometric functions (cf. [17, Chap. XIV]). We should note that in this standard form such Selberg sums can be easily computed as the resultant can be evaluated by a continued-fraction type of recursion. This is implement in, for example, the gp/PARI package. I hope to discuss the results of these calculations in a future paper.

There are some further relations that are of interest. Let n, n' be as in Section 3. If $n'|n$ then the generating series

$$\sum_{\substack{i \equiv i_0 \pmod{n} \\ i \geq 0}} Se(r, \chi_1, \chi_2, i) T^{(i-i_0)/n}$$

which converges in $|T| < q^{-n}$, is, for each $i_0 : 0 \leq i_0 < n$, a rational function in T . There is at most one singularity in $|T| < q^{-n/2}$ and this, if it exists, is simple and is located at $T = q^{-1}(-\tau(\chi_2^{-1}))^{-n}$. These are the series investigated in [10] and [11]. The determination of the residue and the establishing of its properties is one of the major questions addressed in those papers and it is one that has, as yet, only been partially answered.

If the condition $n'|n$ is not satisfied then the series above can also be investigated by means of the theory of Eisenstein series. The point here is that one has to use an Eisenstein series of a “Nebentypus” depending on χ_1 and χ_2 . The general theory is contained in [8]. In fact in this case the constant terms of the Eisenstein series will be made up from holomorphic L-series. The analytic continuation then follows from the “principle of the constant term”; see [9, Theorem 4.8] for a statement and [7, Theorem 1.6.6] for a proof in the function-field case. The details have not been given explicitly but it seems very plausible that in this case one will be able to conclude that the series above also in this case represents a rational function but one now with no singularities in $|T| < q^{-n/2}$. One expects that one will be able to determine the denominator and to estimate the degree of the numerator which are known in the metaplectic case also in the non-metaplectic case.

It is instructive to examine the special case $n = 1$. There are two cases, the metaplectic case in which $\chi_1(r/\cdot)$ is a principal character and the other case when $\chi_1(r/\cdot)$ is non-principal. In the first case we see that $\sum_{i \geq 0} Se(r, \chi_1, \chi_2, i) T^i$ is $L(s, \chi_1(r/\cdot)^{-1})$ with $T = q^{-s}$. Let r_0 be the modulus of $\chi_1(r/\cdot)$; then this series is equal to

$$(1 - q^{1-s}) \prod_{\pi | r_0} (1 - q^{-\deg(\pi)s}).$$

In this case the generating series is a polynomial.

If $\chi_1(r/\cdot)$ is non-principal then, if we suppose for the moment that the $\chi_1(r/\cdot)$ is primitive. Then Anderson, [1], has observed that $L(s, \chi_1(r/\cdot))$ is a polynomial in q^{-s} of degree $\deg(r_0) - 1$. It follows that $L(s, \chi_1(r/\cdot)^{-1})$ has this number of singularities, counted with multiplicities. They lie on $q^{-s} = q^{-1/2}$ by Weil's theorem. If the character is not primitive then there is a numerator of the form $\prod (1 - q^{-s \deg(\pi)})$ where the product is over all the primes dividing the modulus but not the conductor. In the case of the standard Selberg sums the degree of is 1 and it is of the form $1 - \alpha q^{-s}$ where α is a Jacobi sum. The case of other r will be much less straightforward.

We return to the general case. The singularities of the generating series of the sequence $Se(r, \chi_1, \chi_2, i)$ are given as certain sums of their coefficients, i.e. of Selberg sums. This has been described in [10],[11]. The theory of metaplectic groups then leads to a number of relations between the residues, the so-called "Hecke relations". A consequence of this is that there are a number of unexpected relations between Selberg sums. Some examples are given in [10, §4]. To date there is no systematic method of treating these relationships. This is not unlike the situation in the theory of generalized hypergeometric functions.

It is interesting to consider Anderson's method in this connection. His approach is based on a formula which is derived from the theory of Dirichlet series over R . We shall now sketch his technique. Let f be a monic polynomial in R . Let χ be as before and let n be its order. We regard $\chi(f/\cdot)$ as a Dirichlet character. Let f_o be the conductor of f . This is $\prod \pi$ where π runs through those monic prime divisors of f whose order in f is not divisible by n . We shall assume that $f_o \neq 1$. Then by [1, Prop. 2.1]

$$\omega(D(f_o))\chi(f/f_o')\tau(\chi^{-1})^{\deg(f)-1} \prod_{\pi|f_o} \tau(\chi^{-\text{ord}_\pi(f)})^{-\deg(\pi)}$$

is equal to

$$\sum_{\substack{g: \deg(g)=\deg(f_o)-1 \\ g \text{ monic}}} \chi(f/g).$$

This is proved by using the functional equation for the L-function $L(s, \chi(f/\cdot))$ and the fact that the latter is a polynomial to determine the coefficient of $(q^{-s})^{\deg(f_o)-1}$. Anderson proves this in the case $n = q - 1$ but the proof is valid whenever $\chi(f/\cdot)$ is not principal. Anderson achieves the same generality through his parameter c .

Anderson's proof of the formula for the Selberg sum exploits the evaluation of a double sum in two different ways. The argument is strongly

reminiscent of the the multiple Dirichlet series technique, as used, for example, in [3]. The crucial point of the argument is that r_o should be quadratic and it follows that one can also evaluate Selberg sums explicitly when r is a power of an irreducible quadratic polynomial.

It seems plausible that one could use Anderson’s method combined with elementary considerations to determine the properties of metaplectic Eisenstein series over rational function fields. In fact the “principle of the constant term” referred to above is relatively elementary and as the constant term is easy to study in this case even the Eisenstein series approach is relatively elementary. The method of multiple Dirichlet series is, at least for higher ranks, an important component in the study of Eisenstein series and so the applicability of both methods should not be surprising. At any rate it should reassure those unfamiliar with this theory that the method is, at heart, in this case at least, elementary.

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